

S -Noetherian generalized power series rings

F. Padashnik, A. Moussavi and H. Mousavi

Department of Pure Mathematics, Faculty of Mathematical Sciences,
Tarbiat Modares University, Tehran, Iran, P.O. Box: 14115-134.¹

Abstract

Let R be a ring with identity, (M, \leq) a commutative positive strictly ordered monoid and ω_m an automorphism for each $m \in M$. The skew generalized power series ring $R[[M, \omega]]$ is a common generalization of (skew) polynomial rings, (skew) power series rings, (skew) Laurent polynomial rings, (skew) group rings, and Mal'cev Neumann Laurent series rings. If $S \subset R$ is a multiplicative set, then R is called right S -Noetherian, if for each ideal I of R , $Is \subseteq J \subseteq I$ for some $s \in S$ and some finitely generated right ideal J . Unifying and generalizing a number of known results, we study transfers of S -Noetherian property to the ring $R[[M, \omega]]$. We also show that the ring $R[[M, \omega]]$ is left Noetherian if and only if R is left Noetherian and M is finitely generated. Generalizing a result of Anderson and Dumitrescu, we show that, when $S \subseteq R$ is a σ -anti-Archimedean multiplicative set with σ an automorphism of R , then R is right S -Noetherian if and only if the skew polynomial ring $R[x; \sigma]$ is right S -Noetherian.

Keywords: S -Noetherian ring, skew generalized power series ring; right archimedean ring; skew Laurent series ring; skew polynomial ring.

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1 Introduction

Throughout this paper, R is a ring (not necessary commutative) with identity. In [3], the authors introduced the concept of “almost finitely generated” to study Querré’s characterization of divisorial ideals in integrally closed polynomial rings. Later, Anderson and Dumitrescu [1] abstracted this notion to any commutative ring and defined a general concept of Noetherian rings. They call R an S -Noetherian ring if each ideal of R is S -finite, i.e., for each ideal I of R , there exist an $s \in S$ and a finitely generated ideal J of R such that $Is \subseteq J \subseteq I$. By [1, Proposition 2(a)], any integral domain R is $(R \setminus \{0\})$ -Noetherian; so an S -Noetherian ring is not generally Noetherian. Also, M is said to be S -finite if there exist an $s \in S$ and a finitely generated R -submodule F of M such that $sM \subseteq F$. Also, M is called S -Noetherian if each submodule of M is S -finite. In [1], the authors gave a number of S -variants of well-known results for Noetherian rings: S -versions of Cohens result, the Eakin-Nagata theorem, and the Hilbert basis theorem under an additional condition. More precisely, in [1, Propositions 9 and 10], the authors showed that, if S is an anti-Archimedean subset of an S -Noetherian ring R , then the polynomial ring $R[X_1, \dots, X_n]$ is also an S -Noetherian ring; and if S is an anti-Archimedean subset of an S -Noetherian ring R consisting of nonzero divisors, then the power

¹Corresponding author. moussavi.a@modares.ac.ir and moussavi.a@gmail.com.

f.padashnik@modares.ac.ir

h.moosavi@modares.ac.ir .

series ring $R[[X_1, \dots, X_n]]$ is an S -Noetherian ring. Note that if S is a set of units of R , then the results above are nothing but the Hilbert basis theorem and a well-known fact that $R[[X]]$ is Noetherian if R is Noetherian. In [16, Theorem 2.3], Liu generalized this result to the ring of generalized power series as follows: If S is an anti-Archimedean subset of a ring R consisting of nonzero divisors and (Γ, \leq) is a positive strictly ordered monoid (defined in Section 4), then $R[[M, \leq]]$ is S -Noetherian if and only if R is S -Noetherian and Γ is finitely generated. Note that this recovers the result for the Noetherian case shown in [6, Theorem 4.3] when S is a set of units. Also, the authors in [14] study on transfers of the S -Noetherian property to the constructions $D + (X_1, \dots, X_n)E[X_1, \dots, X_n]$ and $D + (X_1, \dots, X_n)E[[X_1, \dots, X_n]]$ and Nagata's idealization is studied in [15].

The authors in [8, Theorem 7.7, page(65)] proved that $R[M]$ is Noetherian if and only if R is Noetherian and M is finitely generated. Brookfield [6] proved that if (M, \leq) is a commutative positively ordered monoid, then $R[[M, \leq]]$ is right Noetherian if and only if R is right Noetherian and M is finitely generated.

Ribenboim [22] and Varadarajan [26], have carried out an extensive study of rings of generalized power series. They investigated conditions under which a ring of generalized power series $R[[M, \leq]]$ is Noetherian, where R is a commutative ring with identity and (M, \leq) is a strictly ordered monoid.

In this paper we obtain results pertaining to Noetherian nature of generalized power series rings. These considerably strengthen earlier results of Ribenboim [22], Varadarajan [26], Brookfield [6], D. D. Anderson, and T. Dumitrescu [1], D. D. Anderson, B. G. Kang, and M. H. Park [2], D. D. Anderson, D. J. Kwak, M. Zafrullah [3] on this topic.

More precisely, we show that, if S is an σ -anti-Archimedean multiplicative subset of an S -Noetherian ring R with an automorphism σ , then the skew polynomial ring $R[x; \sigma]$ is also an S -Noetherian ring; and if (M, \leq) is a commutative positively ordered monoid and ω_m is an automorphism over R for every $m \in M$, then the skew generalized power series ring $R[[M, \omega]]$ is right Noetherian if and only if R is right Noetherian and M is finitely generated. When (M, \leq) is a commutative positive strictly ordered monoid and ω_m is an automorphism for each $m \in M$, we unify and generalize the above mentioned results, and study transfers of S -Noetherian property to the skew generalized power series ring $R[[M, \omega]]$.

2 S-Noetherian property on skew polynomial rings

If R is a commutative ring and S is a multiplicative subset of R , in [1], the authors proved that the necessary condition for the ring of fractions R_S to be a Noetherian ring is that R be an S -Noetherian ring. In noncommutative rings, the situation is more complicated. In fact, if S is a right (resp., left) permutable and right (resp., left) reversible (i.e S is right (resp., left) denominator set), then R has a ring of fraction RS^{-1} (resp., $S^{-1}R$). In this situation, denominator sets (both left and right denominator sets) act like a multiplicatively closed sets in the commutative case. Our interest in this note is multiplicatively closed subsets (i.e. denominator subsets) in noncommutative rings. First we define the notion of S -Noetherian rings for noncommutative rings.

Definition 2.1. Let R be a ring and S a multiplicative subset of R . An ideal I of R is called right S -finite (resp., S -principal), if there exists a finitely generated (resp., principal) right ideal J of R and some $s \in S$ such that $Is \subseteq J \subseteq I$.

A ring R is said to be right S -Noetherian (resp., S -PRIR), if each right ideal of R is right S -finite (resp., S -principal). This definition can be done similarly for left side ideals.

Also, we say that an R -module M is right (or left) S -finite if $Ms \subseteq F$ (resp., $sM \subseteq F$) for some $s \in S$ and a finitely generated submodule F of M . A module M is called right (or left) S -Noetherian if each submodule of M is a right (or left) S -finite module.

The author in [1] justified the definition of S -Noetherian for commutative rings by proving some interesting properties of S -Noetherian ring. For example, they showed that if R is S -Noetherian, then the ring of fractions R_S is Noetherian and they found the conditions for the reverse of this proposition.

Given rings R, T , an ideal J of T is said to be extended, if there exists an ideal I of R such that $\varphi(I) = J$ where $\varphi : R \rightarrow T$ is a ring monomorphism. Also, a ring R is von Neumann regular if for every $a \in R$ there exists an x in R such that $a = axa$. The center of a ring R is denoted by $Cent(R)$.

Proposition 2.2. *Let R be a ring, $S \subseteq R$ a multiplicative set and I a right ideal of R .*

- 1) *If R is von Neumann regular, S a denominator set and $I \cap S \neq \emptyset$, then I is right S -principal.*
- 2) *If $S \subseteq T$ are right denominator subsets of R and R is right S -Noetherian (resp., S -PRIR), then R is right T -Noetherian (resp., T -PRIR).*
- 3) *If R is von Neumann regular and S a denominator set, then R is right S -Noetherian (resp., S -PRIR) if and only if R is right Noetherian (resp., PRIR).*
- 4) *If S is a denominator set and R is right S -Noetherian (resp., S -PRIR), then RS^{-1} is right Noetherian.*
- 5) *If S is central in R , then the conditions 1-4 and those of [1, Proposition 2] follow.*

Proof. 1) Let $S \subseteq R$ be a denominator set, R a von Neumann regular ring and I a right ideal of R . Then for each $s \in I \cap S$, one can see that $Is \subseteq Rs = s_s^{-1}Rs$, where $\frac{1}{s}$ is the inverse of s in RS^{-1} . It is sufficient to see that $\frac{1}{s}Rs \subseteq R$. For each $s \in S$, there exists $a \in R$ such that $sas = s$, so $sa = s_s^{-1} = 1$ (in RS^{-1}). Thus $sa = 1$ and hence $a = \frac{1}{s}$. Therefore $\frac{1}{s} \in R$ and $Rs \subseteq R$, so $\frac{1}{s}Rs \subseteq R$.

2) Let $S \subseteq T$ be denominator subsets of R . If R is right S -Noetherian (resp., S -PRIR), then for each right ideal of R , there exists $s \in S$ such that $Is \subseteq J \subseteq I$ for some finitely generated (resp., principal) right ideal of R . Since $s \in S$, $S \subseteq T$, $s \in T$ which means that R is right T -Noetherian (resp., T -PRIR).

3) Assume that R is a right Noetherian (resp., PRIR) ring. Each right ideal of R is finitely generated (resp., principal). So for each $s \in S$, one can see that $Is \subseteq I$. Hence R is right S -Noetherian (resp., S -PRIR). On the other hand, assume that R is right S -Noetherian (resp., S -PRIR), so there exists $s \in S$ such that $Is \subseteq J \subseteq I$ for some finitely generated (resp., principal) right ideal of R . Also suppose that $sts = s$ for some $t \in R$. So $Is \subseteq I$. Also, $It \subseteq I$, so $Its \subseteq Is = Ists$. So $Its \cdot \frac{1}{s} \subseteq Ists \cdot \frac{1}{s}$. Hence $It \subseteq Ist$. Also $Is \subseteq I$ yields that $Ist \subseteq It \subseteq Ist$. So $Ist = It$. Thus $Ists = Its$ which means that $Is = Ists = Its$. However $Its = I_s^{-1}sts = I_s^{-1}s = I$. So $Is = I$. Thus $I = Is \subseteq J \subseteq I$ and hence $I = J$, and since J is a finitely generated (resp., principal) right ideal of R , so is I .

4) This proof is an inspiration from [4, proposition 3.11 part (i)]. First, we claim that each ideal of RS^{-1} is extended. Let a right ideal J of ring of fraction RS^{-1} and $\frac{x}{s} = b \in J$. So

$\frac{x}{1} = \frac{x}{s} \cdot \frac{s}{1} \in J \cdot \frac{s}{1} \subseteq J$. So $\frac{x}{1} \in J$. Hence, $\varphi^{-1}(\frac{x}{1}) \in \varphi^{-1}(J)$ which means that $x \in \varphi^{-1}(J)$. Thus, $\varphi(x) \in \varphi(\varphi^{-1}(J))$, so $\frac{x}{1} \in \varphi(\varphi^{-1}(J))$. So $\frac{x}{1} \cdot \frac{s}{s} = \frac{x \cdot s}{s \cdot s} = \frac{xs}{s} \in \varphi(\varphi^{-1}(J))$. Note that $\varphi(\varphi^{-1}(J))$ is an ideal of RS^{-1} and $s \in U(RS^{-1})$, so we have

$$\frac{xs}{s} \cdot \frac{1}{s} = \frac{x}{s} \in \varphi(\varphi^{-1}(J)) \frac{1}{s} \subseteq \varphi(\varphi^{-1}(J)).$$

So $b = \frac{x}{s} \in \varphi(\varphi^{-1}(J))$ which implies $J \subseteq \varphi(\varphi^{-1}(J))$. On the other hand, $\varphi(\varphi^{-1}(J)) \subseteq J$ holds for each ideal of RS^{-1} . Thus $J = \varphi(\varphi^{-1}(J))$ and J is an extended ideal of RS^{-1} .

Let a right ideal K of ring of fraction RS^{-1} . Since R is right S -Noetherian there exists $s \in S$ and a finitely generated (resp., principal) right ideal W of R such that $\varphi^{-1}(K)s \subseteq W \subseteq \varphi^{-1}(K)$. So $\varphi(\varphi^{-1}(K)s) \subseteq \varphi(W) \subseteq \varphi(\varphi^{-1}(K))$. We know that $\varphi(\varphi^{-1}(K)s) = \varphi(\varphi^{-1}(K))\varphi(s)$. Also, $\varphi(s) \in U(RS^{-1})$ and $\varphi(\varphi^{-1}(K)) = K$. So $K \subseteq \varphi(W) \subseteq K$. So $K = \varphi(W)$. Since W is finitely generated, $\varphi(W)$ is finitely generated. So K is finitely generated which means that RS^{-1} is right Noetherian.

5) The proof is straightforward by [1, Proposition 2]. \square

Now we generalize a theorem of D.D. Anderson and Tiberiu Dumitrescu [1, Proposition 9], for commutative polynomial ring $R[x]$, in a more general setting. We show that if R is a right (or left) S -Noetherian ring with an automorphism σ , then $R[x; \sigma]$ is a right (or left) S -Noetherian ring.

In [2] the authors defined the notion of anti-Archimedean multiplication set. Now we introduce the notion of σ -anti-Archimedean multiplication set:

Definition 2.3. Let R be a ring with an automorphism σ and S a multiplicative set. Then R is called left σ -anti-Archimedean over S , if there exists $s \in S$, such that

$$\left(\bigcap_{l \geq 1, k_i \geq 0} R\sigma^{k_1}(s)\sigma^{k_2}(s) \cdots \sigma^{k_l}(s) \right) \cap S \neq \emptyset.$$

Theorem 2.4. Let R be a ring with an automorphism σ and $S \subseteq R$ a σ -anti-Archimedean multiplicative set. Then R is right (or left) S -Noetherian if and only if $R[x; \sigma]$ is right (or left) S -Noetherian.

Proof. (\Rightarrow) We prove the theorem for the right version. The proof of left version is similar. First, we claim that if D is a finitely generated R -module and R is a right S -Noetherian ring, then D is a right S -Noetherian module. For this claim, assume that D is a finitely generated right R -module. So there exists a finitely generated free right R -module F and a surjective homomorphism $\pi : F \rightarrow D$. We show that D is a right S -Noetherian R -module. For this, let $N := \pi^{-1}(T)$, for a submodule T of D . We have $N \simeq I_1 \oplus I_2 \cdots \oplus I_l$, for some right ideals I_i of R , $1 \leq i \leq l$. Since R is a right S -Noetherian ring, there exists $s_i \in S$ such that $I_i s_i \subseteq J_i$ for a finitely generated ideals J_i of R , $1 \leq i \leq l$. Now take $s' := s_1 s_2 \cdots s_l \in S$, we show that $N s' \subseteq K$ for a finitely generated R -submodule K of F . One can see that $N s_1 = I_1 s_1 \oplus I_2 s_1 \oplus \cdots \oplus I_l s_1$. Since I_i is a right ideal of R so we have $I_i s_1 \subseteq I_i$ for $i \neq 1$ and $I_1 s_1 \subseteq J_1$, for a finitely generated right ideal J_1 of R . So we have $N s_1 \subseteq J_1 \oplus I_2 \oplus I_3 \cdots \oplus I_l$. Continuing in this way, $N s_1 s_2 \cdots s_l \subseteq J_1 \oplus J_2 \oplus \cdots \oplus J_l \simeq K$, where J_i is a finitely generated right ideal of R , $1 \leq i \leq l$, and hence K is a finitely generated R -submodule of F . Thus $N s' \subseteq K$ and hence F is a right S -Noetherian R -module. Next, since $T = \pi(N)$ and $N s' \subseteq K$, we have $\pi(N s') = \pi(N) s' = T s' \subseteq \pi(K)$. We know that K is finitely generated in F , so $\pi(K)$ is finitely

generated R -submodule of D . Thus, $Ts' \subseteq \pi(K)$ which means that T is S -finite. Since T is an arbitrary R -submodule of D , D is a right S -Noetherian module.

Now, we prove that $A := R[x; \sigma]$ is a right S -Noetherian ring. Let I be right ideal of A and suppose that

$$J = \{r_i \in R \mid r_i \text{ is a leading coefficient of any polynomial in } I\} \cup \{0\}.$$

It is easy to see that J is a right ideal. Since R is right S -Noetherian, $Js \subseteq (a_1R + a_2R + \cdots + a_nR)$ for some $s \in S$ and $a_i \in J$. So there exist polynomials $f_i \in I$ with $f_i = a_{i,n_i}x^{n_i} + \cdots + a_{0,i}$. Let $d = \max\{n_i\}$. Assume that T is the set of all polynomials in I with degree less than d . Obviously, T is a finitely generated right R -submodule of A . So by the first claim, T is right S -Noetherian. Hence there exist $t \in S$, $g_i \in T$ for $1 \leq i \leq m$ such that $Tt \subseteq (g_1R + g_2R + \cdots + g_mR)$. Let $h(x) = \sum_{i=1}^z b_i x^i \in I$, so $b_z \in J$ which means that $b_z \in (a_1R + a_2R + \cdots + a_nR)$. Thus $h\sigma^{-z}(s)$ can be written as follows:

$$h\sigma^{-z}(s) = v^{(1)} + w^{(1)} + q^{(1)},$$

where $v^{(1)} \in (f_1A + f_2A + \cdots + f_nA)$, $w^{(1)} \in \{f \in A \mid d+1 \leq \deg(f) \leq z-1\}$ and $q^{(1)} \in T$. Continuing in this way and multiplying $\sigma^{-z+1}(s), \sigma^{-z+2}(s), \dots, \sigma^{-1-d}(s)$ from right side respectively, so there exists some $v \in (f_1A + f_2A + \cdots + f_nA)$, $w \in T$ such that

$$h\sigma^{-z}(s)\sigma^{-z+1}(s) \cdots \sigma^{-d-1}(s) = v + w.$$

Assume that $s_i = \sigma^{-z+i}$ and multiplying t from right side, then $hs_1s_2 \cdots s_{z-d}t = vt + wt$. But $wt \in Tt$, so $wt \in (g_1R + g_2R + \cdots + g_mR) \subseteq (g_1A + g_2A + \cdots + g_mA)$. Hence,

$$hs_1s_2 \cdots s_{z-d}t \in (f_1A + f_2A + \cdots + f_nA + g_1A + \cdots + g_mA).$$

Since s_i 's and t are independent from the choice of $h \in I$, we have

$$Is_1s_2 \cdots s_{z-d}t \subseteq (f_1A + f_2A + \cdots + f_nA + g_1A + \cdots + g_mA).$$

Finally, since $s_1s_2 \cdots s_{z-d}t \in S$, the ideal I is S -finite and because I was chosen an arbitrary right ideal of A , hence A is a right S -Noetherian ring.

(\Leftarrow) Let I be a right ideal of R . Suppose that

$$J = \{f \in A \mid \text{the leading coefficient of } f \text{ is in } I\}.$$

Then J is a right ideal of A . Since A is right S -Noetherian, there exists $s \in S$ such that $Js \subseteq K \subseteq J$, where K is a finitely generated right ideal of A . Suppose that $K = (f_1A + f_2A + \cdots + f_lA)$. Let $r \in I$, then there exists some $f \in J$ such that $fs = \sum a_i f_i$. So if r_i is the leading coefficient of f_i , $1 \leq i \leq l$, then $rs \in (r_1R + r_2R + \cdots + r_lR)$. So $Is \subseteq (r_1R + r_2R + \cdots + r_lR)$. Also, $K \subseteq J$, so each leading coefficient of K is a leading coefficient of J . So $(r_1R + r_2R + \cdots + r_lR) \subseteq I$ and hence I is right S -finite and R is right S -Noetherian. \square

We have the following generalization of a theorem of D.D. Anderson and Tiberiu Dumitrescu [1, Proposition 9].

Corollary 2.5. *Let R be a (not necessarily commutative) ring and $S \subseteq R$ an anti-Archimedean multiplicative set. If R is S -Noetherian then so is the polynomial ring $R[X_1, X_2, \dots, X_n]$.*

3 Noetherian Skew Generalized Power Series Rings

Throughout this section, (M, \leq) is assumed to be a strictly ordered commutative monoid. The pair (M, \leq) is called an *ordered monoid* with order \leq , if for every $m, m', n \in M$, $m \leq m'$ implies that $nm \leq nm'$ and $mn \leq m'n$. Also, an ordered monoid (M, \leq) is said to be *strictly ordered* if for every $m, m', n \in M$, $m < m'$ implies that $nm < nm'$ and $mn < m'n$. Let (M, \leq) be a partially ordered set. The set (M, \leq) is called *Artinian* if every strictly decreasing sequence of elements of M stabilizes, and also (M, \leq) is called *narrow* if the number of incomparable elements in every subset of M is finite. Thus, we can conclude that (M, \leq) is Artinian and narrow if and only if every nonempty subset of M has at least one but only a finite number of minimal elements.

The author in [24] introduced the ring of generalized power series $R[[M]]$ for a strictly ordered monoid M and a ring R consisting of all functions from M to R whose support is Artinian and narrow with the pointwise addition and the convolution multiplication. There are a lot of interesting examples of rings in this form (e.g., Elliott and Ribenboim, [7]; Ribenboim, [23]) and it was extensively studied by many authors, recently.

In [21], the authors defined a “twisted” version of the mentioned construction and study on ascending chain condition for its principal ideals. Now we recall the construction of the skew generalized power series ring introduced in [21]. Let R be a ring, (M, \leq) a strictly ordered monoid, and $\omega : M \rightarrow \text{End}(R)$ a monoid homomorphism. For $m \in M$, let ω_m denote the image of m under ω , that is $\omega_m = \omega(m)$. Let A be the set of all functions $f : M \rightarrow R$ such that the support $\text{supp}(f) = \{m \in M \mid f(m) \neq 0\}$ is Artinian and narrow. Then for any $m \in M$ and $f, g \in A$ the set

$$\chi_m(f, g) = \{(u, v) \in \text{supp}(f) \times \text{supp}(g) : m = uv\}$$

is finite. Thus one can define the product $fg : M \rightarrow R$ of $f, g \in A$ as follows:

$$fg(m) = \sum_{(u,v) \in \chi_m(f,g)} f(u)\omega_u(g(v)),$$

(by convention, a sum over the empty set is 0). Now, the set A with pointwise addition and the defined multiplication is a ring, and called *the ring of skew generalized power series* with coefficients in R and exponents in M . To simplify, take A as a formal series $\sum_{m \in M} r_m x^m$, where

$r_m = f(m) \in R$. This ring can be denoted either by $R[[M^{\leq}, \omega]]$ or by $R[[M, \omega]]$ (see [18] and [19]).

For every $r \in R$ and $m \in M$ we can define the maps $c_r, e_m : M \rightarrow R$ by

$$c_r(x) = \begin{cases} r & ; x = 1 \\ 0 & ; \text{Otherwise} \end{cases}, e_m(x) = \begin{cases} 1 & ; x = m \\ 0 & ; \text{Otherwise} \end{cases} \quad (3.1)$$

where $x \in M$. By way of illustration, $c_r(x)$ and $e_m(x)$ are like r and x^m in usual polynomial ring $R[x]$, respectively.

The following proposition which is proved in [11, Theorem 2.1], can characterize all Artinian and narrow sets.

Proposition 3.1. *Let (M, \leq) be an ordered set. Then the following conditions are equivalent*

- (1) (M, \leq) is Artinian and narrow.
- (2) For any sequence $(m_n)_{n \in \mathbb{N}}$ of elements of M there exist indices $n_1 < n_2 < n_3 < \dots$ such that $m_{n_1} \leq m_{n_2} \leq m_{n_3} \leq \dots$.
- (3) For any sequence $(m_n)_{n \in \mathbb{N}}$ of elements of M there exist indices $i < j$ such that $m_i \leq m_j$.

The author in [6] introduced the concept of a lower set. A *lower set* of L is a subset $I \subseteq L$ such that $x \leq y \in I$ implies $x \in I$ for all $x, y \in L$, (which we denoted by $\Downarrow L$ for the set of lower sets of L ordered by inclusion). In this concept, we can ignore the condition narrow by lower set, indeed it is proved that if L is a partially ordered set, then $\Downarrow L$ is Artinian if and only if L is Artinian and narrow. He also showed that if $\alpha : K \rightarrow L$ is strictly increasing map between partially ordered sets, then if L satisfies Artinian (or Noetherian) property, then so is K . Moreover, if α is surjective and $\Downarrow K$ satisfies Artinian (or Noetherian) property, then so does $\Downarrow L$.

An ordered monoid (M, \leq) is called *positively ordered* if $m \geq 0$ for all $m \in M$. In this condition, $m \preceq m'$ implies $m \leq m'$ for all $m, m' \in M$. Now, according to [6, in section 4] we have

$$R[[M, \omega, \leq]] = \{f \in R[[M, \omega]] \mid \Downarrow (\text{supp}(f), \leq) \text{ is Artinian}\}. \quad (3.2)$$

If $\Downarrow (M, \leq)$ is Artinian, $R[[M, \omega, \leq]] = R[[M, \omega]]$. For instance, $\Downarrow (\mathbb{F}, \preceq)$ and $\Downarrow (\mathbb{F}^n, \preceq)$ are Artinian, and so $R[[\mathbb{F}, \omega, \preceq]] = R[[\mathbb{F}, \omega]]$ and $R[[\mathbb{F}^n, \omega, \preceq]] = R[[\mathbb{F}^n, \omega]]$ such that \mathbb{F} be a free monoid.

Now we give a generalization of a result [6, Theorem 4.3] of G. Brookfield:

Theorem 3.2. *Let R be a ring, (M, \leq) a positive strictly ordered monoid and ω_m an automorphism of R with $\omega_m \omega_n = \omega_n \omega_m$ for each $m, n \in M$. Then $R[[M, \omega]]$ is left Noetherian if and only if R is left Noetherian and M is finitely generated.*

Proof. \Leftarrow) In the first place, we claim that if $\varphi : (N, \leq) \rightarrow (M, \leq)$ is a surjective strict monoid homomorphism, induces a surjective ring homomorphism $\varphi^* : R[[N, \omega, \leq]] \rightarrow R[[M, \omega, \leq]]$. Since φ is strict, $\varphi^{-1}(x)$ is antichain in (N, \leq) for all $x \in M$. Thus, if $f \in R[[N, \omega, \leq]]$ then $\varphi^{-1}(x) \cap \text{supp}(f)$ is finite and we can define $\varphi^*(f) = f^*$, where $f^*(x) = \sum_{x' \in \varphi^{-1}(x)} f(x')$ for $x \in M$. We show that φ^* is a ring homomorphism. One can see that

$$(fg)^*(m) = \sum_{m' \in \varphi^{-1}(m)} (fg)(m') = \sum_{xy=m} \sum_{\substack{x'y'=m' \\ m' \in \varphi^{-1}(m)}} \left(f(x') \alpha_{x'}(g(y')) \right). \quad (3.3)$$

On the other hand

$$\begin{aligned} (f^*g^*)(m) &= (\varphi^*(f)\varphi^*(g))(m) = \sum_{xy=m} \left(\varphi^*(f(x)) \alpha_x(\varphi^*(g(y))) \right) \\ &= \sum_{xy=m} \left(\sum_{x' \in \varphi^{-1}(x)} f(x') \right) \alpha_x \left(\sum_{y' \in \varphi^{-1}(y)} g(y') \right) \\ &= \sum_{xy=m} \sum_{x' \in \varphi^{-1}(x)} \sum_{y' \in \varphi^{-1}(y)} \left(f(x') \alpha_{x'}(g(y')) \right). \end{aligned}$$

Since φ^{-1} is a homomorphism, $\varphi^{-1}(x)\varphi^{-1}(y) = \varphi^{-1}(xy)$ and so $\varphi^{-1}(m) = \varphi^{-1}(x)\varphi^{-1}(y)$. So

$$(f^*g^*)(m) = \sum_{xy=m} \sum_{\substack{m'=x'y' \\ m' \in \varphi^{-1}(m)}} \left(f(x')\alpha_{x'}(g(y')) \right). \quad (3.4)$$

By equations 3.3 and 3.4 we see that $(fg)^*(m) = (f^*g^*)(m)$. We have also

$$\begin{aligned} (f+g)^*(x) &= \sum_{x' \in \varphi^{-1}(x)} (f+g)(x') = \sum_{x' \in \varphi^{-1}(x)} (f(x') + g(x')) \\ &= \sum_{x' \in \varphi^{-1}(x)} f(x') + \sum_{x' \in \varphi^{-1}(x)} g(x') = f^*(x) + g^*(x). \end{aligned}$$

Thus $\varphi^* : R[[N, \omega, \leq]] \rightarrow R[[M, \omega, \leq]]$ is a ring homomorphism. Now, we show that φ^* is surjective. Suppose that $f \in R[[M, \omega, \leq]]$, where $\{f(n)\}_{n \in M}$ are the coefficients of f in R . For every $n \in M$, the set $\varphi^{-1}(n)$ is nonempty and finite, say $\varphi^{-1}(n) = \{m_1, m_2, \dots, m_k\}$, where k and all the m_i depends on n . We define the function $g \in R[[N, \omega, \leq]]$ as follows

$$g(m_j) = \begin{cases} f(n) & ; j = 1 \\ 0 & ; \text{otherwise.} \end{cases} \quad (3.5)$$

Notice that g is independent of n , since if $n \neq n'$, then $\varphi^{-1}(n) \cap \varphi^{-1}(n') = \emptyset$. Also, for each $n \in M$ we have

$$\varphi^*(g)(n) = \sum_{m \in \varphi^{-1}(n)} g(m) = \sum_{j=1}^k g(m_j) = g(m_1) = f(n).$$

This means that $\varphi^*(g) = f$, and hence φ^* is surjective. So we proved the claim. It is well-known that there is an strict monoid surjection $\varphi : (\mathbb{F}^n, \preccurlyeq) \rightarrow (M, \preccurlyeq)$ for some $n \in \mathbb{N}$. Also, the identity map $(M, \preccurlyeq) \rightarrow (M, \leq)$ is a surjection. So the composition of these two maps is a surjection and by [6, Lemma 2.1]. Hence $R[[M, \omega, \leq]]$ is a homomorphic image of the ring $R[[\mathbb{F}^n, \omega, \preccurlyeq]]$. Since $R[[\mathbb{F}^n, \omega, \preccurlyeq]] = R[[\mathbb{F}^n, \omega]]$ and $R[[\mathbb{F}^n, \omega]]$ is Noetherian, its projection $R[[M, \omega, \leq]]$ is also Noetherian. Moreover, we show that $R[[M, \omega, \leq]] = R[[M, \omega]]$. If $R[[M, \omega, \leq]]$ is left Noetherian, then $\downarrow (M, \preccurlyeq)$ is Artinian. By applying [6, Lemma 2.1(2)] to the identity map $(M, \preccurlyeq) \rightarrow (M, \leq)$, one can see that $\downarrow (M, \leq)$ is Artinian. Thus $R[[M, \omega, \leq]] = R[[M, \omega]]$.

\Rightarrow) The method of this part is inspired from [6, Theorem 4.3]. The trivial case of M is obvious. By [6, Lemmas 3.1 and 3.2], M is strict and \preccurlyeq is a partial order on M .

Suppose $T = R[[M, \omega, \leq]]$ is left Noetherian. One can see that M is finitely generated similar to the proof of [6, Theorem 4.3]. Hence we have to prove that R is Noetherian similar to the proof of ([25, Theorem 5.2(i)], [26, Theorem 3.1(i)]). Let $I_T = \{f \in T \mid \omega_x(f(y)) \in I; x, y \in M\}$. It is easy to see that I_T is a left ideal of T . So for each ideal I of R , there is a correspondent ideal in T . Also if $I \subset J$, then $I_T \subset J_T$. Hence if there exists a nonstabilized ascending chain in R , then there is one in T . But this is impossible, so R is left Noetherian. \square

In Theorem 3.2 if we set σ the identity homomorphism then we have:

Corollary 3.3. [6, Theorem 4.3] *Let R be a ring and (M, \leq) a positive strictly ordered monoid. Then $R[[M, \leq]]$ is left Noetherian if and only if R is left Noetherian and M is finitely generated.*

Finally, we conclude the following result which connects the results of previous sections.

Corollary 3.4. *Let R be an S -Noetherian von Neumann regular ring and S a denominator set. Assume that (M, \leq) is a finitely generated positive strictly ordered monoid and ω_m an automorphism of R with $\omega_m \omega_n = \omega_n \omega_m$ for each $m, n \in M$. Then $(S^{-1}R)[[M, \omega]]$ is a left Noetherian ring.*

Proof. The ring $S^{-1}R$ is Noetherian by Theorem 2.2. Since (M, \leq) is a positive strictly ordered monoid and ω_m is an automorphism for all $m \in M$, $(S^{-1}R)[[M, \omega]]$ is a Noetherian ring by Theorem 3.2. \square

4 S-Noetherian property of generalized skew power series rings

Recall that a ring is called right *duo* (resp., left duo) if all of its right (resp., left) ideals are two-sided. Also, a right and left duo ring is called a duo ring. We know that if a ring is duo, then every prime ideal is completely prime. It is known that a power series ring over a duo ring need not be duo (on either side).

Lemma 4.1. *Let R be a duo ring and $S \subset R$ a denominator set. If $s \in S$, $r \in R$ then there exists $s_1 \in S$ such that $srs_1 = rss_1$.*

Proof. Let $s \in S$ and $r \in R$. Since R is duo, there exist $s' \in S$ such that $sr = rs'$, so $\frac{1}{s} \cdot \frac{sr}{1} = \frac{1}{s} \cdot \frac{rs'}{1}$. Hence $\frac{r}{1} = \frac{rs'}{s} = \frac{r}{1} \cdot \frac{s'}{s}$. Thus $\frac{r}{1}(1 - \frac{s'}{s}) = 0$, which means that $\frac{r(s-s')}{s} = 0_{S^{-1}R}$. So $r(s-s')s_1 = 0_R$. So $rss_1 = rs's_1$ and since $rs' = sr$ we have $srs_1 = rss_1$. \square

In the previous result, it is easy to see that if $s \in S$, $r \in R$, then there exists $s_1 \in S$ such that $s_1sr = s_1rs$. We will use this point in the proposition below.

Proposition 4.2. *Let R be a duo ring, $S \subseteq R$ a denominator set and M an S -finite R -module. Then M is S -Noetherian if and only if PM is an S -finite submodule, for each S -disjoint prime ideal P of R .*

Proof. The “only if” part is clear. For the converse, assume that PM is S -finite for each P prime ideal of R with $P \cap S = \emptyset$. Since M is S -finite, $wM \subseteq F$ for some $w \in S$ and some finitely generated submodule F . If M is not S -Noetherian, the set \mathfrak{F} of all non- S -finite submodules of M is not empty. So \mathfrak{F} has a maximal element like N by Zorn’s lemma. We claim that $P = [N : M] := \{r \in R \mid rM \subseteq N\}$ is a prime ideal of R and is disjoint from S . Suppose to the contrary that $P \cap S \neq \emptyset$ and $s \in P \cap S$. Then we have

$$swN \subseteq swM \subseteq sF \subseteq sM \subseteq N.$$

So $swN \subseteq sF \subseteq N$ and N becomes S -finite. This contradiction shows that $P \cap S = \emptyset$. Now suppose that P is not a prime ideal of R . So P is not completely prime. So there exist $a, b \in R \setminus P$ and $ab \in P$. So $N + aM$ is S -finite, hence $s(N + aM) \subseteq (R(n_1 + am_1) + \cdots + R(n_p + am_p))$ for some $s \in S$, $n_i \in N$ and $m_i \in M$. Also $[N : a]$ is S -finite. So $t[N : a] \subseteq (Rq_1 + Rq_2 + \cdots + Rq_k)$ for some $t \in S$ and $q_j \in [N : a]$. Since R is duo and S is a denominator set in R , there exists $s'' \in S$ such that $s''at = s''ta$ by Theorem 4.1. Also $s(N + aM) \subseteq (R(n_1 + am_1) + \cdots + R(n_p + am_p))$. Thus $sx = \sum r_i n_i + r_i a m_i$. This means that $sx = \sum r_i n_i + a \sum r'_i m_i$ for some $r'_i \in R$. Since $sx, \sum r_i n_i \in N$, we have $\sum r'_i m_i \in [N : a]$. So

$$s''tsx = s''t \sum r_i n_i + s''t \sum ar'_i m_i = \sum s''tr_i n_i + s''at \sum r'_i m_i = \sum s''tr_i n_i + s''a \sum c_j q_j.$$

So $s''tsx = \sum s''tr_in_i + \sum c's''_jaq_j$ for some $c'_j \in R$. Hence $s''tsx \in (Rn_1 + \cdots + Rn_p + Rs''aq_1 + \cdots + Rs''aq_k)$. So $s''tsN \subseteq (Rn_1 + \cdots + Rn_p + Rs''aq_1 + \cdots + Rs''aq_k) \subseteq N$. Thus N is S -finite and this contradicts to the fact that N is maximal in \mathfrak{F} . Therefore P is a prime ideal of R . Moreover $P = [N : M] \subseteq [N : F] \subseteq [N : wM] = [P : w] = P$. Hence $[N : F] = P$. Let $F = (Rf_1 + Rf_2 + \cdots + Rf_k)$. Since R is a duo ring, $P = [N : \sum Rf_i] = \bigcap [N : f_i]$. So $P = [N : f_i]$ for some $f_i \in \{f_1, f_2, \dots, f_k\}$. One can show that $tN \subseteq (Rn_1 + Rn_2 + \cdots + Rn_l) + PM$ for some $t \in S$ and $n_i \in N$ as above or in similar way as that employed in [1, Proposition 4]. Since PM is S -finite, $vPM \subseteq G \subseteq PM \subseteq N$ for some $v \in S$ and a finitely generated submodule G of M . So

$$vtN \subseteq v(Rn_1 + Rn_2 \cdots + Rn_l) + vPM \subseteq (Rn'_1 + Rn'_2 \cdots + Rn'_l) + G \subseteq N$$

for some $n'_i \in N$. So N becomes S -finite which is a contradiction. So M is S -Noetherian. \square

Lemma 4.3. *Let R be a ring with an endomorphism σ . If $R[[x; \sigma]]$ is a duo ring, then σ is surjective.*

Proof. Suppose that $a \in R$. Since $R[[x; \sigma]]$ is a duo ring we have $ax = xf$ such that $f = \sum_{i=0}^{\infty} f_i x^i$. So $xf = x \sum_{i=0}^{\infty} f_i x^i = \sum_{i=0}^{\infty} \sigma(f_i) x^{i+1}$. Now, since $ax = xf$, $\sigma(f_i) = 0$ for all $i \neq 0$ and $\sigma(f_0) = a$. Thus, for each $a \in R$ there exists $f_0 \in R$ such that $a = \sigma(f_0)$. \square

Theorem 4.4. *Let R be a ring, $S \subseteq R$ a σ -anti-Archimedean denominator set (consisting nonzero divisors) and $\sigma_1, \dots, \sigma_n$ are monomorphisms of R with $\sigma_i \sigma_j = \sigma_j \sigma_i$, for each i, j . Assume that $R[[X_1, \dots, X_n; \sigma_1, \dots, \sigma_n]]$ is a duo ring. If R is S -Noetherian, then the ring $R[[X_1, \dots, X_n; \sigma_1, \dots, \sigma_n]]$ is also S -Noetherian.*

Proof. We use the method in [1, Proposition 10] employed by Anderson and Dumitrescu. As S is σ -anti-Archimedean in every ring containing R as a subring, we shall prove the case $n = 1$, so we assume that $T = R[[x; \sigma]]$ is duo and σ is an automorphism of R . It is enough to prove that every prime ideal P of T is S -finite. Let $\pi : T \rightarrow R$ the R -algebra homomorphism sending x to zero and $P' = \pi(P)$. Since R is S -Noetherian, there exists $s \in S$ such that $sP' \subseteq (Rg_1(0) + Rg_2(0) + \cdots + Rg_k(0))$ for some $g_i \in P$. If $x \in P$, then $P = (TP' + Tx)$. If $g_i(x) = \sum a_i x^i$, then $g_i(x) = \sum x^i \sigma^{-i}(a_i) \in (TP' + Tx)$. So $sP \subseteq (TP' + Tx) = (Tg_1 + \cdots + Tg_k) \subseteq P$. This means that P is S -finite. Let $x \notin P$ and $f \in P$. So $sf(0) = \sum d_{0,j} g_j(0)$ for some $d_{0,j} \in R$. So $xf_1 = sf - \sum d_{0,j} g_j \in P$ for some $f_1 \in T$. Considering $x \notin P$, $f_1 \in P$. So $sf_1 = \sum d_{1,j} g_j + xf_2$ for some $f_2 \in T$. Hence $\sigma(s)sf = \sum \sigma(s)d_{0,j} g_j + x \sum d_{1,j} g_j + x^2 f_2$. Also $f_2 \in P$, since $x \notin P$ and $sf_1 - \sum d_{1,j} g_j \in P$. In this way, one can see that for each $L \geq 0$,

$$\left(\prod_{l=0}^L \sigma^l(s) \right) f = \sum_{i=0}^L x^i \sum_{j=1}^k \left(\prod_{l=i+1}^L \sigma^l(s) \right) d_{i,j} g_j + x^{L+1} f_{L+1}.$$

Since $S \cap \left(\bigcap_{l \geq 1, i_j \in \mathbb{N} \cup \{0\}} \sigma^{i_1}(s) \cdots \sigma^{i_l}(s) R \right) \neq \emptyset$, there exists $t \in R$ such that $\frac{t}{\sigma^{i_1}(s) \cdots \sigma^{i_k}(s)} \in R$ for each $i_j \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$. Moreover

$$tf = \sum_j \sum_i \left(\frac{t \sigma^{-i}(d_{i,j})}{\prod_l \sigma^l(s)} \right) x^i g_j.$$

So $tf = \sum_j h_j g_j$ where $h_j = \sum_i \frac{t \sigma^{-i}(d_{i,j})}{\prod_l \sigma^l(s)} x^i$. So $tf \in (Tg_1 + Tg_2 + \cdots + Tg_k)$. Hence $tP \subseteq (Tg_1 + Tg_2 + \cdots + Tg_k)$. Since $g_i \in P$, $(Tg_1 + Tg_2 + \cdots + Tg_k) \subseteq P$. Thus $R[[x; \sigma]]$ is an S -Noetherian ring. \square

The following proposition which is proved in [1], is the corollary of the above theorem.

Corollary 4.5. [1, Proposition 10] *Let R be a commutative ring and $S \subseteq R$ an anti-Archimedean multiplicative set of R . If R is S -Noetherian, then so is $R[[X_1, \dots, X_n]]$.*

A ring R is called *strongly regular* if every principal right (or left) ideal is generated by a central idempotent. A ring is said to be *left self injective* if it is injective as a left module over itself. Hirano in [12, Theorem 4] shows that if R is a self-injective strongly regular ring, then $R[[x]]$ is a duo ring.

We have the following generalization of a theorem of D.D. Anderson and Tiberiu Dumitrescu [1, Proposition 10].

Theorem 4.6. *Let R be a duo ring with an automorphism σ and $S \subseteq R$ a σ -anti-Archimedean denominator set (consisting nonzero divisors). If R is S -Noetherian, then so is the skew power series ring $R[[x; \sigma]]$.*

Proof. We can prove this theorem in a similar way as in Theorem 4.4. Consider the notations in the proof of Theorem 4.4. Let $x \in P$. Since σ is bijective, P is S -finite. Let $x \notin P$ and $f \in P$, so $xf_1 = sf - \sum d_{0i}g_i \in P$. Note that for each $h \in R[[x; \sigma]]$ and I is a left ideal of $R[[x; \sigma]]$, $xh \in I$ yields that $xR[[x; \sigma]]h \in I$. So $f_1 \in P$. The rest of the proof is similar to what we did in Theorem 4.4. \square

The following corollary is a generalization of the case $n = 1$ in [1, Proposition 10] for the category of duo rings.

Corollary 4.7. *Let R be a duo ring and $S \subseteq R$ an anti-Archimedean denominator set (consisting nonzero divisors) of R . If R is S -Noetherian, then so is the power series ring $R[[x]]$.*

Now we extend the last result for the skew generalized power series ring $R[[M, \omega]]$.

Theorem 4.8. *Let R be a duo ring, (M, \leq) a positive strictly ordered commutative monoid and ω_m a monomorphism of R with $\omega_m \omega_n = \omega_n \omega_m$ for each $m, n \in M$. Assume that $S \subset R$ is an ω_m -anti-Archimedean denominator set (consisting nonzero divisors) of R and $R[[M, \omega]]$ be a duo ring. Then $R[[M, \omega]]$ is left (or right) S -Noetherian if and only if R is left (or right) S -Noetherian and M is finitely generated.*

Proof. (\Leftarrow) We use the method of G. Brookfield employed in [6]. We know that the surjective homomorphism $\varphi : \mathbb{F}^n \rightarrow M$ (where \mathbb{F} is a free monoid) induces a projection

$$\varphi^* : R[[\mathbb{F}^n, (\omega, \preceq)]] \rightarrow R[[M, (\omega, \leq)]]$$

and $R[[M, (\omega, \leq)]] = R[[M, \omega]]$ by [6, Theorem 4.3]. Moreover, since $R[[\mathbb{F}^n, (\omega, \preceq)]]$ is S -Noetherian, so is $R[[M, \omega]]$ by [16, Lemma 2.2] for noncommutative version.

(\Rightarrow) Let $A := R[[M, \omega]]$ be S -Noetherian. Let $\{m_n | n \in \mathbb{N}\}$ be an infinite sequence in M . Let $I = (Ae_{m_1} + Ae_{m_2} + \dots)$. Since A is S -Noetherian, there exists $s \in S$ such that $c_s I \subseteq J \subseteq I$ for J finitely generated ideal of A . So $c_s I \subseteq (Ae_{m_{i_1}} + Ae_{m_{i_2}} + \dots + Ae_{m_{i_k}})$ for some $k \in \mathbb{N}$. So $c_s e_{m_l} = \sum_{t=0}^k f_t e_{m_{i_t}}$ for some $l \neq i_t$. So $m_l \in \bigcup_{t=0}^k \text{supp}(f_t e_{m_{i_t}})$ for each $m \in M$, $(f_t e_{m_{i_t}})(m) = \sum_{m' m'' = m} f_t(m') \omega_{m'}(e_{m_{i_t}}(m''))$. So $m_l \in \bigcup_{t=0}^k \{\text{supp}(f_t) + \text{supp}(\omega_{m'}(e_{m_{i_t}}(m'')))\}$. There exists $m_1 \in M$ such that $m_1 m_{i_t} = m$ for some $0 \leq t \leq L$. So

$$(f_t e_{m_{i_t}})(m) = f_t(m_1) \omega_{m_1}(e_{m_{i_t}}(m_{i_t})) = f_t(m_1).$$

Thus $m_1 \in \text{supp}(f_t)$ and $m_1 m_{i_t} \in \text{supp}(f_t e_{m_{i_t}})$ for some $0 \leq t \leq L$. So for each $m \in \text{supp}(\omega_{m'}(e_{m_{i_t}}(m')))$, $m_{i_t} \preceq m$ for some $0 \leq t \leq L$. Since $m_l \in \text{supp}(f_t e_{m_{i_t}})$, $m_{i_t} \preceq m_l$ for some $0 \leq t \leq L$. Since M is positive strictly ordered monoid, M is finitely generated by [6, Lemma 3.3].

Let I be an ideal of R , so AI is an ideal of A . So there exists $s \in S$ such that $c_s AI \subseteq J \subseteq AI$ for some J finitely generated ideal of A . Set

$$T = \{f(\pi(f)) | f \in c_s AI\}.$$

We claim that $T = sI$. Let $t \in T$, so $t = h(\pi(h))$ and $h = c_s g$ for some $g \in AI$. So $t = sg(\pi(sg))$. This means that $t \in sI$ considering the fact that

$$I = \{f(\pi(f)) | f \in AI\}.$$

So $T \subseteq sI$. Now let $i \in I$, so $i \in AI$. Since $si(m) = 0$ for $m \neq 1$, $si \in T$. Thus $sI \subseteq T$. Hence $sI = T$. But $sI = T \subseteq J' \subseteq I$ where $J' = \{f(\pi(f)) | f \in J\}$. Let $J = (Aj_1 + Aj_2 + \cdots + Aj_p)$. So it is easy to show that $J' = (Rj_1(\pi(j_1)) + Rj_2(\pi(j_2)) + \cdots + Rj_p(\pi(j_p)))$. So J' is finitely generated in R . Hence I is S -finite and R is left S -Noetherian. \square

Recall from [5], that a ring R is right (left) \aleph_0 -injective provided any homomorphism from a countably generated right (left) ideal of R into R extends to a right (left) R -module endomorphism of R . By an \aleph_0 -injective ring we mean a right and left \aleph_0 -injective ring.

Corollary 4.9. *Let R be an strongly regular and an \aleph_0 -injective ring with automorphisms σ_1, σ_2 such that $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$. Assume that $S \subset R$ is an σ_1, σ_2 anti-Archimedean denominator set (consisting nonzero devisors). If R is left (or right) S -Noetherian, then so is $R[[x, y; \sigma_1, \sigma_2]]$.*

Proof. Assume that R is an strongly regular and \aleph_0 -injective ring. Then by [20], $A = R[[x; \sigma_1]]$ is duo ring and S -left Noetherian ring. Then $A[[y; \sigma_2]]$ is left S -Noetherian ring. \square

The following corollary is a generalization of the case $n = 2$ of in [1, Proposition 10] for the category of duo rings.

Corollary 4.10. *Let R be an strongly regular self-injective ring and $S \subseteq R$ an anti-Archimedean denominator set (consisting nonzero devisors) of R . If R is left (or right) S -Noetherian, then so is $R[[x, y]]$.*

Corollary 4.11. *Let R be a duo ring, $S \subseteq R$ an anti-Archimedean denominator set (consisting nonzero devisors) of R . Assume that $R[[M]]$ is a duo ring. Then $R[[M]]$ is left (or right) S -Noetherian if and only if R is left (or right) S -Noetherian and M is finite generated.*

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